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REGULAR CHARACTERS OF GROUPS OF TYPE A_n OVER DISCRETE VALUATION RINGS

ROI KRAKOVSKI, URI ONN, AND POOJA SINGLA

ABSTRACT. Let \mathfrak{o} be a complete discrete valuation ring with finite residue field k of odd characteristic. Let \mathbf{G} be a general or special linear group or a unitary group defined over \mathfrak{o} and let \mathfrak{g} denote its Lie algebra. For every positive integer ℓ , let K^ℓ be the ℓ -th principal congruence subgroup of $\mathbf{G}(\mathfrak{o})$. A continuous irreducible representation of $\mathbf{G}(\mathfrak{o})$ is called regular of level ℓ if it is trivial on $K^{\ell+1}$ and its restriction to $K^\ell/K^{\ell+1} \simeq \mathfrak{g}(k)$ consists of characters with $\mathbf{G}(k)$ -stabiliser of minimal dimension. In this paper we construct the regular characters of $\mathbf{G}(\mathfrak{o})$, compute their degrees and show that the latter satisfy Ennola duality. We give explicit uniform formulae for the regular part of the representation zeta functions of these groups.

1. INTRODUCTION

1.1. Overview. Let F be a non-archimedean local field with ring of integers \mathfrak{o} , maximal ideal \mathfrak{p} and finite residue field k of cardinality q and characteristic p . Thus F is either the field $k((t))$ of formal Laurent series in one variable over k or a finite extension of the field \mathbb{Q}_p of p -adic numbers. For a positive integer n let \mathbf{G} be either the general linear group \mathbf{GL}_n or the standard unitary group \mathbf{GU}_n attached to an unramified quadratic extension E/F .

In this paper we construct a family of continuous complex-valued irreducible representations of the groups $\mathbf{G}(\mathfrak{o})$ and $\mathbf{G}'(\mathfrak{o}) = \mathbf{G}(\mathfrak{o}) \cap \mathbf{SL}_n(E)$ and analyse their dimensions. Being a maximal compact subgroup of $\mathbf{G}(F)$, the group $\mathbf{G}(\mathfrak{o})$ and its representations play an important role in the construction of admissible representations of $\mathbf{G}(F)$; see, for instance, [3]. In a different direction, representations of the groups $\mathbf{G}'(\mathfrak{o})$ play a crucial role in the representation theory of arithmetic groups of type A_{n-1} . Indeed, suppose that \mathcal{O} is the ring of integers in a global field L and that \mathbf{H} is a connected, simply-connected absolutely almost simple algebraic group of type A_{n-1} defined over \mathcal{O} . Assume that $\mathbf{H}(\mathcal{O})$ has the congruence subgroup property, which means that its finite quotients are exhausted by the quotients of $\mathbf{H}(\mathcal{O}/I)$, where I ranges over the non-zero ideals of \mathcal{O} . Then, the finite dimensional irreducible complex representations of $\mathbf{H}(\mathcal{O})$ are finite tensor products of representations of the completions $\mathbf{H}(\mathcal{O}_v)$, v a finite place of L , and rational representations of $\mathbf{H}(\mathbb{C})$ when L is a number field; see [11]. For all but finitely many places these completions are of the form $\mathbf{G}'(\mathfrak{o})$.

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In contrast to the well-understood representation theory of the finite groups of Lie type $\mathbf{G}(k)$ or of the locally compact groups $\mathbf{G}(F)$, representations of $\mathbf{G}(\mathfrak{o})$ are considerably less understood; see, for example, [2], [14], [15], [17] and [13] for analysis of cases with small rank or small level; see also [16] for a comprehensive account of the state of the art for the case of general linear groups.

The focus of this paper is on regular representations of $\mathbf{G}(\mathfrak{o})$, which we now discuss. Recall that by virtue of the profiniteness of $\mathbf{G}(\mathfrak{o})$, every complex continuous irreducible representation ρ of $\mathbf{G}(\mathfrak{o})$ has a *level* $\ell = \ell(\rho)$; this is the smallest natural number ℓ such that ρ factors through the principal congruence quotient $\mathbf{G}(\mathfrak{o}/\mathfrak{p}^{\ell+1})$. Let \mathfrak{g} denote the Lie algebra scheme of \mathbf{G} and, for every positive integer m , let $K^m = \text{Ker}(\mathbf{G}(\mathfrak{o}) \rightarrow \mathbf{G}(\mathfrak{o}/\mathfrak{p}^m))$ be the m -th principal congruence subgroup of $\mathbf{G}(\mathfrak{o})$. There are $\mathbf{G}(k)$ -equivariant isomorphisms $K^m/K^{m+1} \simeq (\mathfrak{g}(k), +)$ and a $\mathbf{G}(k)$ -equivariant isomorphism $x \mapsto \varphi_x$ between $\mathfrak{g}(k)$ and its Pontryagin dual $\mathfrak{g}(k)^\vee = \text{Hom}_{\mathbb{Z}}(\mathfrak{g}(k), \mathbb{C}^\times)$; see Section 2.3.

Definition 1.1. (Regular representations and characters) A character $\varphi_x \in \mathfrak{g}(k)^\vee$ is called *regular* if x is a regular element of $\mathfrak{g}(k)$, that is, the characteristic polynomial of x is equal to its minimal polynomial. An irreducible representation of level ℓ of $\mathbf{G}(\mathfrak{o})$ and its character are called *regular* if the orbit of their restriction to K^ℓ consists of pullbacks of regular characters of $K^\ell/K^{\ell+1} \simeq \mathfrak{g}(k)$.

Regular elements of $\mathbf{G}(\bar{k})$, where \bar{k} is an algebraic closure of k , are defined as elements whose centraliser has minimal dimension, see e.g. [19, §3.5]. We extend this definition to the present situation and consider the adjoint action of $\mathbf{G}(k)$ on $\mathfrak{g}(k)$. In this case the stabilisers of regular elements have minimal dimension, namely, the rank of the group $n = \text{rank}(\mathbf{G})$; they are abelian; and the corresponding matrix is cyclic. See [6, Theorem 3.6] for a proof of these equivalences over \bar{k} and deduce them for k using base change.

1.2. Main results. Our first main result is a construction of the regular characters of $\mathbf{G}(\mathfrak{o})$ using explicit characters of their principal congruence subquotients.

Theorem A. *Let \mathbf{G} be \mathbf{GL}_n or \mathbf{GU}_n and let \mathfrak{o} be a complete discrete valuation ring with finite residue field of odd characteristic. Let $\ell > 1$ be an integer and let $m = \lfloor \ell/2 \rfloor$. For every regular character χ of $\mathbf{G}(\mathfrak{o})$ of level $\ell - 1$ there exists an explicitly constructed character σ of K^m/K^ℓ such that σ extends to its stabiliser in $\mathbf{G}(\mathfrak{o})$ and the extended character induces irreducibly to give χ .*

The study of regular characters for the groups $\mathbf{GL}_n(\mathfrak{o})$ was initiated by Hill in [6, 7]. Hill constructed regular characters associated to split regular matrices in $\mathfrak{gl}_n(k)$. Takase noted in [20] that Hill's argument fails in the non-semisimple case and proposed a uniform method to construct the regular characters, which relies on a conjectural vanishing of certain Schur multipliers [20, Conjecture 4.6.5]. He proved this conjecture in the semisimple case and obtained an unconditional construction in that case. Although the present paper follows a somewhat different path, we encounter the same multiplier and show that (a more general form of) the conjecture holds; see Theorem 3.1 and Remark 3.7. Regular characters of $\mathbf{GL}_n(\mathfrak{o})$ that

correspond to irreducible matrices in $\mathfrak{gl}_n(k)$ were also constructed in [1]. Recently, Stasinski and Stevens announced a construction of the regular characters of $\mathbf{GL}_n(\mathfrak{o})$, see [16, 18]. Their construction follows a different approach than the one taken here and applies to even residual characteristic as well.

Our next result gives uniform formulae for the degrees of the regular characters. We call a matrix $\tau \in M_n(\mathbb{Z}_{\geq 0})$ an n -type if $n = \sum_{d,e} d\tau_{d,e}$. Let \mathcal{A}_n be the set of n -types. It turns out (see Section 4) that regular elements in $\mathfrak{g}(k)$ can be partitioned according to n -types and the latter determine the former's centralisers and eventually also determine the degrees of the corresponding regular characters. For $\epsilon \in \{\pm 1\}$ and $\tau \in \mathcal{A}_n$ define the following polynomials in $\mathbb{Z}[x]$

$$v_\epsilon(x) = x^{n^2} \prod_{1 \leq d \leq n} (1 - \epsilon^d x^{-d}), \text{ and}$$

$$u_\epsilon^\tau(x) = x^n \prod_{1 \leq d, e \leq n} (1 - \epsilon^d x^{-d})^{\tau_{d,e}}.$$

Let

$$\varepsilon_{\mathbf{G}} = \begin{cases} 1 & \text{if } \mathbf{G} = \mathbf{GL}_n, \\ -1 & \text{if } \mathbf{G} = \mathbf{GU}_n. \end{cases}$$

We remark that $v_{\varepsilon_{\mathbf{G}}}(q)$ is the order of $\mathbf{G}(k)$ and $u_{\varepsilon_{\mathbf{G}}}^\tau(q)$ is the order of the centraliser in $\mathbf{G}(k)$ of a regular element in $\mathfrak{g}(k)$ of type τ ; see Proposition 4.4.

Theorem B. *Let \mathbf{G} be either \mathbf{GL}_n or \mathbf{GU}_n and let $\varepsilon_{\mathbf{G}}$ be $+1$ or -1 , respectively. Then, for every complete discrete valuation ring \mathfrak{o} with finite residue field of cardinality $q > n$ and odd characteristic, the degrees of the regular characters of $\mathbf{G}(\mathfrak{o})$ of level ℓ are*

$$q^{\binom{n}{2}(\ell-1)} v_{\varepsilon_{\mathbf{G}}}(q) / u_{\varepsilon_{\mathbf{G}}}^\tau(q), \quad \tau \in \mathcal{A}_n.$$

In particular, the degrees of the regular characters satisfy Ennola duality: for every polynomial $f \in \mathbb{Z}[x]$ the evaluation $f(q)$ is the degree of a regular character of $\mathbf{GL}_n(\mathfrak{o})$ if and only if $(-1)^{\deg(f)} f(-q)$ is the degree of a regular character of $\mathbf{GU}_n(\mathfrak{o})$ of the same level.

We remark that the character degrees provided by Theorem B form a superset of the possible degrees for $q \leq n$ as well, but in such case not all of them occur. The reminiscence in Theorem B of 'Ennola duality' gives further evidence that this phenomenon features in the groups $\mathbf{GL}_n(\mathfrak{o})$ and $\mathbf{GU}_n(\mathfrak{o})$ in general. In [5] Ennola observed that characters of $\mathbf{GL}_n(\mathfrak{o}/\mathfrak{p})$ and of $\mathbf{GU}_n(\mathfrak{o}/\mathfrak{p})$ are related upon a formal substitution of ' $-q$ ', see also [21]. In fact, all characters of groups of type A_1 and A_2 obey this duality, as observed in [2], and also characters of level one of groups of type A_3 , as observed in [12].

The next result gives the number of regular characters of type τ and level ℓ . For $d \in \mathbb{N}$, let

$$(1.1) \quad w_d(x) = \frac{1}{d} \sum_{m|d} \mu(d/m) x^m \in \mathbb{Q}[x],$$

where μ is the Möbius function. Thus $w_d(q)$ is the number of monic irreducible polynomials of degree d over k . Let $\binom{a}{b}$ and $\binom{a}{b_1, b_2, \dots, b_r}$ denote the binomial and multinomial coefficients, respectively. In what follows all summations and products indexed by d and e range over the set $\{1, \dots, n\}$.

Theorem C. *Let \mathbf{G} be either \mathbf{GL}_n or \mathbf{GU}_n and let $\varepsilon_{\mathbf{G}}$ be $+1$ or -1 , respectively. Then, for every complete discrete valuation ring \mathfrak{o} with finite residue field of cardinality $q > n$ and odd characteristic, the number of regular characters of $\mathbf{G}(\mathfrak{o})$ of type $\tau \in \mathcal{A}_n$ and level $\ell \in \mathbb{N}$ is*

$$q^{(\ell-1)n} u_{\varepsilon_{\mathbf{G}}}^{\tau}(q) \prod_d \binom{\sum_e \tau_{d,e}}{\tau_{d,1}, \tau_{d,2}, \dots} \binom{w_d(q)}{\sum_e \tau_{d,e}}.$$

A group Γ is called *rigid* if for every $m \in \mathbb{N}$ there is a finite number, denoted $r_m(\Gamma)$, of inequivalent complex m -dimensional irreducible representations of Γ . The group Γ has *polynomial representation growth* if the sequence $r_m(\Gamma)$ is bounded by a polynomial in m . One then defines the representation zeta function of Γ to be the Dirichlet generating function $\zeta_{\Gamma}(s) = \sum_m r_m(\Gamma) m^{-s}$. If Γ is topological one counts only continuous representations. The group $\mathbf{G}(\mathfrak{o})$ is not rigid; for instance, there are infinitely many linear continuous characters factoring through the determinant map. However, the number of regular representations of $\mathbf{G}(\mathfrak{o})$ of dimension m is bounded by polynomial in m . Following Stasinski¹ we look at the regular part of the representation zeta function of the groups $\mathbf{G}(\mathfrak{o})$, the Dirichlet generating function that counts only regular characters. Combining Theorems B and C gives an explicit computation of the regular zeta function.

Corollary D. *The regular part of the representation zeta function of $\mathbf{G}(\mathfrak{o})$ is*

$$\zeta_{\mathbf{G}(\mathfrak{o})}^{\text{reg}}(s) = \frac{1}{1 - q^{n - \binom{n}{2}s}} \sum_{\tau \in \mathcal{A}_n} u_{\varepsilon_{\mathbf{G}}}^{\tau}(q) \prod_d \binom{\sum_e \tau_{d,e}}{\tau_{d,1}, \tau_{d,2}, \dots} \binom{w_d(q)}{\sum_e \tau_{d,e}} \left(\frac{v_{\varepsilon_{\mathbf{G}}}(q)}{u_{\varepsilon_{\mathbf{G}}}^{\tau}(q)} \right)^{-s}.$$

Finally, we give an explicit computation of the regular representation zeta function associated with $\mathbf{G}'(\mathfrak{o})$. For $\tau \in \mathcal{A}_n$ and $r \in \mathbb{N}$ let

$$(1.2) \quad \iota(\tau, r) = \gcd(\{e \mid \exists d \text{ with } \tau_{d,e} \neq 0\} \cup \{r\}).$$

Theorem E. *Let \mathbf{G}' be either \mathbf{SL}_n or \mathbf{SU}_n and let $\varepsilon_{\mathbf{G}}$ be $+1$ or -1 , respectively. Then, for every discrete valuation ring \mathfrak{o} with residue field of odd characteristic prime to n and cardinality $q > n$, the regular part of the representation zeta function of $\mathbf{G}'(\mathfrak{o})$ is*

$$\begin{aligned} \zeta_{\mathbf{G}'(\mathfrak{o})}^{\text{reg}}(s) &= \frac{1}{1 - q^{n-1 - \binom{n}{2}s}} \sum_{\tau \in \mathcal{A}_n} \frac{\iota(\tau, q - \varepsilon_{\mathbf{G}})^2}{(q - \varepsilon_{\mathbf{G}})} u_{\varepsilon_{\mathbf{G}}}^{\tau}(q) \\ &\quad \times q^{-1} \prod_d \binom{\sum_e \tau_{d,e}}{\tau_{d,1}, \tau_{d,2}, \dots} \binom{w_d(q)}{\sum_e \tau_{d,e}} \left(\frac{v_{\varepsilon_{\mathbf{G}}}(q)}{u_{\varepsilon_{\mathbf{G}}}^{\tau}(q) \iota(\tau, q - \varepsilon_{\mathbf{G}})} \right)^{-s}. \end{aligned}$$

¹personal communication.

A remarkable feature of the enumerative and quantitative results above, compactly expressed by the various zeta functions, is their uniformity. When \mathbf{G} is one of the forms \mathbf{GL}_n or \mathbf{GU}_n the zeta function depends only on the form (through $\varepsilon_{\mathbf{G}}$) and on the cardinality q of the residue field, where the latter dependence is merely through substitution of q in a finite explicit list of polynomials. In the case of \mathbf{SL}_n or \mathbf{SU}_n one still gets uniform behaviour with some additional sensitivity to divisors of $q - \varepsilon_{\mathbf{G}}$.

We also note that for large n most representations of the groups above are not regular when counted according to dimension. Indeed, the abscissa of convergence of the regular part of the representation zeta function of $\mathbf{G}'(\mathfrak{o})$ given in Theorem E is $2/n$. Nevertheless, by [11, Theorem 8.1] there is a universal lower bound for the abscissae of convergence of the representation zeta functions of open compact subgroups in isotropic groups.

1.3. Organisation of the paper. Section 2 contains notation and preliminary remarks. In Section 3 we prove an elaboration of Theorem A. In Section 4 we describe regular elements in $\mathfrak{g}(k)$ and their centralisers. In Section 5 we prove the remaining results mentioned in the introduction.

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2. NOTATION AND PRELIMINARY REMARKS

For a group G and $g, h \in G$ we denote the group commutator $ghg^{-1}h^{-1}$ by (g, h) . For a Lie ring L we use $[a, b]$ to denote the Lie bracket of $a, b \in L$. All our Lie rings are contained in matrix rings, in particular, the Lie bracket is induced from the associative structure, that is $[a, b] = ab - ba$.

2.1. Rings and fields. The rings and fields that we consider fit into the following diagram

$$\begin{array}{ccccccc} E & \supset & \mathfrak{O} & \triangleright & \mathfrak{P} = \pi\mathfrak{O} & & k_2 \\ | & & | & & | & & | \\ F & \supset & \mathfrak{o} & \triangleright & \mathfrak{p} = \pi\mathfrak{o} & & k \end{array}$$

Here F is non-archimedean local field with ring of integers \mathfrak{o} , maximal ideal \mathfrak{p} , residue field k and a fixed uniformiser π . The field E is an unramified quadratic extension of F with ring of integers \mathfrak{O} . We let $\nu \in \text{Gal}(E/F)$ be the nontrivial Galois automorphism. The residue fields have characteristic p which is assumed to be odd (except in Section 4). For $\ell \in \mathbb{N}$ we let $\mathfrak{o}_\ell = \mathfrak{o}/\mathfrak{p}^\ell$ and similarly $\mathfrak{O}_\ell = \mathfrak{O}/\mathfrak{P}^\ell$ be the finite quotients. For $m < \ell$ we have natural reduction maps $M_n(\mathfrak{O}_\ell) \rightarrow M_n(\mathfrak{O}_m)$ which we denote by $x \mapsto x_m$.

2.2. Groups and Lie algebras. Let \mathbf{G} denote either the general linear group or the following unitary group associated to the extension E/F . Let $\circ : M_n(E) \rightarrow M_n(E)$ be the involution on $n \times n$ matrices defined by $(a_{ij})^\circ \mapsto (a_{ji}^\vee)$. Then \mathbf{GU}_n is the affine algebraic group defined by the equations $x^\circ x = I$. Let \mathfrak{g} be the Lie algebra scheme of \mathbf{G} . Then \mathfrak{g} is either \mathfrak{gl}_n or the anti-hermitian Lie algebra \mathfrak{gu}_n defined by the equations $x + x^\circ = 0$. Let \mathbf{G}' and \mathfrak{g}' stand for $\mathbf{G} \cap \mathbf{SL}_n$ and $\mathfrak{g} \cap \mathfrak{sl}_n$, respectively.

2.3. Duality for Lie rings. The equivariant duality between $\mathfrak{g}(\mathfrak{o}_\ell)$ and its Pontraigin dual $\mathfrak{g}(\mathfrak{o}_\ell)^\vee := \text{Hom}_{\mathbb{Z}}(\mathfrak{g}(\mathfrak{o}_\ell), \mathbb{C}^\times)$ is established as follows. We fix a non-trivial additive character

$$(2.1) \quad \psi : E \rightarrow \mathbb{C}^\times$$

with $\text{Ker}(\psi) = \mathfrak{D}$, see e.g. [2, Lemma 5.12 and Remark 5.13]. We get a well-defined map

$$M_n(\mathfrak{D}_\ell) \rightarrow M_n(\mathfrak{D}_\ell)^\vee, \quad x \mapsto \varphi_x, \text{ where } \varphi_x(y) = \psi(\pi^{-\ell} \text{Tr}(xy)).$$

Moreover, as $\mathfrak{P}^{-1} \not\subset \text{Ker}(\psi)$, this map is an isomorphism and in fact restricts to an isomorphism $\mathfrak{g}(\mathfrak{o}_\ell) \simeq \mathfrak{g}(\mathfrak{o}_\ell)^\vee$ which is $\mathbf{G}(\mathfrak{o}_\ell)$ -equivariant with respect to the adjoint and co-adjoint actions, see [2, Lemma 5.12]. We call a character φ_x *regular* if the corresponding matrix x is regular (cyclic). Note that the notion of regularity is independent of the level, namely, $x \in M_n(\mathfrak{D}_\ell)$ is a cyclic matrix if and only if any of its reductions modulo π^m is cyclic. In such case $C_{M_n(\mathfrak{D}_\ell)}(x) = \mathfrak{D}_\ell[x]$, namely, the centraliser of x is the \mathfrak{D}_ℓ -algebra generated by x .

2.4. Representations and characters. All the groups we consider are compact or moreover finite. We let $\text{Irr}(G)$ denote the set of characters of continuous irreducible representations of G . As our groups are totally disconnected we have, in particular, that $\text{Irr}(G) = \bigcup \text{Irr}(G/N)$, where the union is over the finite index normal subgroups of G . If N is a normal finite index subgroup of G and $\sigma \in \text{Irr}(N)$ we write $I_G(\sigma)$ for the inertia group of σ in G . We write $\text{Irr}(G \mid \sigma)$ for the subset of irreducible characters of G whose restriction to N contains σ .

3. CONSTRUCTION OF THE REGULAR REPRESENTATIONS OF $\mathbf{G}(\mathfrak{o})$

Throughout this section it will be convenient to descend to finite quotients and discuss representations of $\mathbf{G}(\mathfrak{o}_\ell)$ rather than representations of $\mathbf{G}(\mathfrak{o})$ of level $\ell - 1$. For $1 \leq m < \ell$ let $K_\ell^m = K^m/K^\ell = \text{Ker}(\mathbf{G}(\mathfrak{o}_\ell) \rightarrow \mathbf{G}(\mathfrak{o}_m))$. The main result in this section is the following theorem, which together with 3.2 gives an elaborate version of Theorem A.

Theorem 3.1. *Let \mathbf{G} be either \mathbf{GL}_n or \mathbf{GU}_n and let $\mathfrak{g} = \text{Lie}(\mathbf{G})$. Let \mathfrak{o} be a complete discrete valuation ring with finite residue field of odd characteristic. Let $\ell > 1$ be an integer and set $m = \lfloor \ell/2 \rfloor$. Let σ be an irreducible character of K_ℓ^m lying above a regular character of $K_\ell^{\ell-1}$. Then the following hold.*

- (1) *The character σ extends to its inertia group $I_{\mathbf{G}(\mathfrak{o}_\ell)}(\sigma)$. In particular, every such extension induces to a regular character of $\mathbf{G}(\mathfrak{o}_\ell)$.*
- (2) *The character σ corresponds to an orbit of a regular matrix $x \in \mathfrak{g}(\mathfrak{o}_{\ell-m})$ and there exists a bijection*

$$\text{Irr}(\mathbf{G}(\mathfrak{o}_\ell) \mid \sigma) \simeq C_{\mathbf{G}(\mathfrak{o}_m)}(x_m)^\vee,$$

where x_m is the image of x in $\mathfrak{g}(\mathfrak{o}_m)$.

In particular, every regular representation of $\mathbf{G}(\mathfrak{o})$ arises in this manner for appropriate ℓ and σ .

The proof of Theorem 3.1 generalises and modelled after [10, Section 7]. In *loc. cit.* Jaikin-Zapirain analyses the irreducible characters of $\mathbf{SL}_2(\mathfrak{o})$ of positive level which in this case are all regular. We require some preparations recorded in Sections 3.1, 3.2, 3.3 and 3.4 that allow us to prove Theorem 3.1 in Section 3.5.

3.1. Exponent and logarithm. If \mathfrak{o} has characteristic zero, the exponent and logarithm defined by the power series $\exp(x) = \sum_{r=0}^{\infty} x^r/r!$ and $\log(1+x) = \sum_{r=1}^{\infty} (-1)^{r-1} x^r/r$, respectively, define mutually inverse homeomorphisms between K^m and $\mathfrak{g}(\pi^m \mathfrak{o})$. Such a bijection fails in positive characteristic. Nevertheless, by controlling the ratio ℓ/m one obtains a bijection of finite subquotients in positive characteristic as well. Indeed, \exp and \log truncate to polynomials of degree $\lfloor (\ell-1)/m \rfloor$ and establish a bijection $K_{\ell}^m \simeq \mathfrak{g}(\pi^m \mathfrak{o}_{\ell})$ whenever $p = \text{char}(k) > \lfloor (\ell-1)/m \rfloor$.

Two important special cases arise. For $\ell/2 \leq m < \ell$ the exponential map reduces to a linear polynomial $\exp(x) = 1+x$ giving rise to an isomorphism of abelian groups $\exp : \mathfrak{g}(\pi^m \mathfrak{o}_{\ell}) \rightarrow K_{\ell}^m$. If $\ell/3 \leq m < \ell/2$, the exponential map reduces to a quadratic polynomial $\exp(x) = 1+x+\frac{1}{2}x^2$, establishing a bijection $\mathfrak{g}(\pi^m \mathfrak{o}_{\ell}) \rightarrow K_{\ell}^m$. Here enters the main reason for our assumption throughout that $p > 2$, though not the only one. In such case the bijection is no longer an isomorphism of groups but is not 'too far' from being an isomorphism in the sense made precise in the following Lemma (see [10, Section 7]).

Lemma 3.2. *Let ℓ and m be positive integers such that $\ell/3 \leq m < \ell$ and let (M, \star) denote the $\mathbf{G}(\mathfrak{o}_{\ell})$ -module defined as follows. As a set $M = K_{\ell}^m$. The abelian group structure is given by $a \star b := ab(a, b)^{-1/2}$ for all $a, b \in M$ and the $\mathbf{G}(\mathfrak{o}_{\ell})$ -action is the conjugation action. Then $\exp : (\mathfrak{g}(\pi^m \mathfrak{o}_{\ell}), +) \rightarrow (M, \star)$ is an isomorphism of $\mathbf{G}(\mathfrak{o}_{\ell})$ -modules.*

Proof. Whenever \exp and \log converge they are mutual inverses. Therefore, our assumption on m, ℓ and the characteristic of $\mathfrak{o}/\mathfrak{p}$ guarantees that they establish a bijection between $M_n(\pi^m \mathfrak{o}_{\ell})$ and $1 + M_n(\pi^m \mathfrak{o}_{\ell})$. Furthermore, since \exp and \log commute with the involution \circ they restrict to a bijection $\mathfrak{g}(\pi^m \mathfrak{o}_{\ell}) \simeq K_{\ell}^m$. Our assumption on the characteristic also implies that squaring is an automorphism of the subgroup $(K_{\ell}^m, K_{\ell}^m) \subset K_{\ell}^{2m} \simeq (\mathfrak{g}(\mathfrak{o}_{\ell-2m}), +)$ so that the formula for \star is well-defined. The commutativity of \star follows from $(1 + \pi^m y) \star (1 + \pi^m z) = 1 + \pi^m(y + z) + \frac{1}{2}\pi^{2m}(yz + zy)$ modulo π^{ℓ} . Finally, conjugation commutes with \exp and \log , therefore, this is an isomorphism of $\mathbf{G}(\mathfrak{o}_{\ell})$ -modules where the action factors through $\mathbf{G}(\mathfrak{o}_{\ell-m})$. \square

3.2. Characters of K_{ℓ}^m . If $\ell = 2m$ then $K_{\ell}^m \simeq \mathfrak{g}(\pi^m \mathfrak{o}_{\ell})$, consequently, the irreducible characters of K_{ℓ}^m are linear characters of the form $\log^* \theta$ for $\theta \in \mathfrak{g}(\pi^m \mathfrak{o}_{\ell})^{\vee}$. If $\ell = 2m + 1$ the group K_{ℓ}^m is a p -group of nilpotency class two and its irreducible characters are well understood from various perspectives. We describe it in terms of the present setup, as depicted in Figure 1.

FIGURE 1.

$$\begin{array}{ccccc}
 \mathfrak{g}(k) & \longleftarrow & \mathfrak{g}(\pi^m \mathfrak{o}_\ell) & \xleftarrow{\log} & K_\ell^m \\
 | & & | & & | \\
 \bar{\mathfrak{j}} & \longleftarrow & \mathfrak{j} & \xleftarrow{\log} & J \\
 | & & | & & | \\
 \bar{\mathfrak{r}}_{\bar{x}} & \longleftarrow & \mathfrak{r}_{\bar{x}} & \xleftarrow{\log} & R_{\bar{x}} \\
 | & & | & & | \\
 (0) & \longleftarrow & \mathfrak{g}(\pi^{m+1} \mathfrak{o}_\ell) & \xleftarrow{\log} & K_\ell^{m+1}
 \end{array}$$

The subgroup $K_\ell^{m+1} \subset K_\ell^m$ is central and each of its characters are of the form $\log^* \theta$ for some $\theta \in \mathfrak{g}(\pi^{m+1} \mathfrak{o}_\ell)^\vee$. Anticipating the sequel it is convenient to think of θ as the restriction of φ_x for some $x \in \mathfrak{g}(\mathfrak{o}_\ell)$. Let

$$B_\theta : K_\ell^m / K_\ell^{m+1} \times K_\ell^m / K_\ell^{m+1} \rightarrow \mathbb{C}^\times; \quad B_\theta(a K_\ell^{m+1}, b K_\ell^{m+1}) = \theta(\log(a, b)).$$

Then B_θ is an alternating bilinear form on $K_\ell^m / K_\ell^{m+1} \simeq \mathfrak{g}(k)$ with values in \mathbb{C}^\times rather than k . Unravelling definitions it can be described in terms of the alternating bilinear form $\beta_{\bar{x}}(\bar{y}, \bar{z}) = \text{Tr}(\bar{x}[\bar{y}, \bar{z}])$ that makes the diagram

$$\begin{array}{ccc}
 K_\ell^m / K_\ell^{m+1} & \times & K_\ell^m / K_\ell^{m+1} \xrightarrow{B_\theta} \mathbb{C}^\times \\
 | \wr & & | \wr \quad \uparrow \psi(\pi^{-1}(\cdot)) \\
 \mathfrak{g}(k) & \times & \mathfrak{g}(k) \xrightarrow{\beta_{\bar{x}}} k
 \end{array}$$

commute, where $\bar{x} \in \mathfrak{g}(k)$, the reduction of x modulo π , represents the restriction of θ to $\mathfrak{g}(\pi^{\ell-1} \mathfrak{o}_\ell)$. In particular B_θ depends only on \bar{x} . Let $\bar{\mathfrak{r}}_{\bar{x}}$ denote the radical of $\beta_{\bar{x}}$. Then

$$\begin{aligned}
 \bar{\mathfrak{r}}_{\bar{x}} &= \{\bar{y} \in \mathfrak{g}(k) \mid \text{Tr}(\bar{x}[\bar{y}, \bar{z}]) = 0, \forall \bar{z} \in \mathfrak{g}(k)\} && \text{(by definition)} \\
 &= C_{\mathfrak{g}(k)}(\bar{x}) && \text{(by the non-degeneracy of the trace)} \\
 &= \mathfrak{r}_{\bar{x}} / \mathfrak{g}(\pi^{m+1} \mathfrak{o}_\ell) \simeq R_{\bar{x}} / K_\ell^{m+1},
 \end{aligned}$$

where $\mathfrak{r}_{\bar{x}} \subset \mathfrak{g}(\pi^m \mathfrak{o}_\ell)$ is the inverse image of $\bar{\mathfrak{r}}_{\bar{x}}$ under reduction modulo π^{m+1} and $R_{\bar{x}} = \exp(\mathfrak{r}_{\bar{x}})$. The character θ can be extended to $\mathfrak{r}_{\bar{x}}$ in $|\bar{\mathfrak{r}}_{\bar{x}}|$ -many ways. Each such extension θ' determines uniquely an irreducible character of K_ℓ^m that arises as follows. The induced bilinear form on $\mathfrak{g}(k) / \bar{\mathfrak{r}}_{\bar{x}}$ is non-degenerate. Let $\bar{\mathfrak{j}} \supset \bar{\mathfrak{r}}_{\bar{x}}$ be a maximal isotropic subspace and let \mathfrak{j} be its inverse image in $\mathfrak{g}(\pi^m \mathfrak{o}_\ell)$. Let θ'' be an extension of θ' to \mathfrak{j} . Then

- (i) The pullback $\log^* \theta''$ is a character of $J = \exp(\mathfrak{j})$.
- (ii) The character $\sigma = \text{Ind}_J^{K_\ell^m}(\log^* \theta'')$ is irreducible and independent of \mathfrak{j} and θ'' .
- (iii) There is a bijection between extensions θ' of θ to $\mathfrak{r}_{\bar{x}}$ and $\text{Irr}(K_\ell^m \mid \theta)$.

These are well known assertions coined as 'Heisenberg lift'; see [8]. We remark that (i) follows from

$$\log(ab) = \log(a \star b) + \log(a, b)^{1/2} = \log(a) + \log(b) + \log(a, b)^{1/2}, \quad a, b \in K_\ell^m,$$

where the first equality follows from the definition of \star and the fact that $(a, b)^{1/2}$ is central; the second equality follows from the fact that \log is an isomorphism of modules, see Lemma 3.2. Applying θ'' and restricting a, b further down to be in J we get

$$[\theta'' \circ \log](ab) = [\theta'' \circ \log](a) \cdot [\theta'' \circ \log](b),$$

since $\log(a, b)^{1/2}$ is in the kernel of θ .

3.3. Stabilisers in $G(\mathfrak{o}_\ell)$ of regular characters. We now recall the structure of stabilisers of regular characters which have a particularly simple description. This is recorded in the following Lemma, the proof of which in the GL_n case is found in [6, Corollary 3.9]. The same proof works verbatim for the unitary case and is omitted.

Lemma 3.3. *Let $x \in \mathfrak{g}(\mathfrak{o}_\ell)$ be a regular element and let $\varphi = \varphi_x \in \mathfrak{g}(\mathfrak{o}_\ell)^\vee$ be the corresponding regular character. Let $r < \ell/2$ be a positive integer, let $\varphi|_{\ell-r}$ denote the restriction of φ to $\mathfrak{g}(\pi^{\ell-r}\mathfrak{o}_\ell)$ and let $\log^*(\varphi|_{\ell-r})$ be its pullback to $K_\ell^{\ell-r}$ along the log map. Then*

$$I_{G(\mathfrak{o}_\ell)}(\log^*(\varphi|_{\ell-r})) = \text{Stab}_{G(\mathfrak{o}_\ell)}(x_{\ell-r}) = C_{G(\mathfrak{o}_\ell)}(x)K_\ell^r.$$

Lemma 3.3 reveals the main feature of regular characters that simplifies their analysis; their stabilisers can be described in terms of a fixed group, namely the centraliser of a matrix $C_{G(\mathfrak{o}_\ell)}(x) = \mathfrak{O}_\ell[x] \cap G(\mathfrak{o}_\ell)$, together with a principal congruence subgroup.

3.4. Stable maximal isotropic subspace. We require the following standard lemma (stated as Lemma 2.4 in [10] and referenced to the proof of [8, Lemma 1.4]). For completeness we give a simple proof.

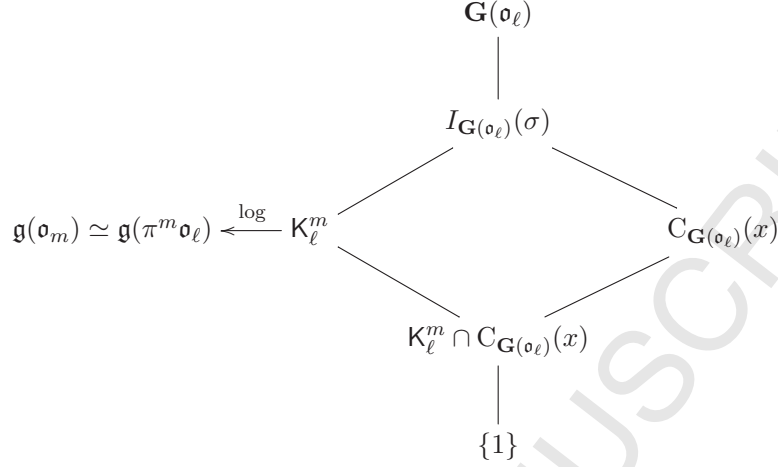
Lemma 3.4. *Let V be a finite dimensional \mathbb{F}_p -vector space and $\alpha : V \times V \rightarrow \mathbb{F}_p$ an antisymmetric bilinear form on V . Suppose that a p -group P is acting on V and preserving α . Then there exists a maximal isotropic subspace U of V which is P -invariant.*

Proof. We use induction on $\dim V$. If $\dim V = 1$ then α is the zero form and there is nothing to prove. Since $|V|$ and $|P|$ are powers of p and $0 \in V$ is a fixed point for P , there is a nonzero P -fixed vector $v \in V$. Let W be the subspace of all vectors in V that are orthogonal to v . Since $v \in W$ we may look at the quotient $\bar{W} := W/\mathbb{F}_p v$. The form α descends to a form $\bar{\alpha}$ on \bar{W} and is preserved by the induced P -action. By induction there is a maximal isotropic P -stable subspace \bar{U} in \bar{W} . Its inverse image in V is the required subspace for (V, α) . \square

3.5. Proof of Theorem 3.1. The proof for the case $\ell = 2m$ is straightforward. Let σ be an irreducible character of K_ℓ^m lying above a regular character of $K_\ell^{\ell-1}$, that is, σ is the pullback under \log of a restriction of $\varphi = \varphi_x$ for a regular element x in $\mathfrak{g}(\mathfrak{o}_\ell)$.

Lemma 3.3 guarantees that $I_{G(\mathfrak{o}_\ell)}(\sigma) = C_{G(\mathfrak{o}_\ell)}(x)K_\ell^m$ is a product of abelian groups, where $C_{G(\mathfrak{o}_\ell)}(x)$ normalises K_ℓ^m and stabilises σ . The various subgroups are depicted in Figure 2.

FIGURE 2. Even case $\ell = 2m$



It follows that every character of $C_{G(o_\ell)}(x)$ that agrees with σ on $K_\ell^m \cap C_{G(o_\ell)}(x)$ glues with σ to a linear character of $I_{G(o_\ell)}(\sigma)$ that extends σ . By Clifford theory every such extension induces irreducibly to $G(o_\ell)$, proving the first assertion. The number of such extensions is $I_{G(o_\ell)}(\sigma)/K_\ell^m \simeq C_{G(o_m)}(x_m)$, and the second assertion follows.

Assume now that $\ell = 2m + 1$. We refer the reader to Figure 3 which features all the main players of the construction: linear algebra over k , Lie theory over \mathfrak{o} and group theory.

Let σ be an irreducible character of K_ℓ^m lying above a regular linear character $\log^* \theta$ of K_ℓ^{m+1} , for $\theta \in \mathfrak{g}(\pi^{m+1} o_\ell)^\vee$, such that $\theta|_{\mathfrak{g}(\pi^{\ell-1} o_\ell)}$ corresponds to a regular element $\bar{x} \in \mathfrak{g}(k)$. We claim that there exists a lift $x \in \mathfrak{g}(o_\ell)$ of \bar{x} such that

$$(3.1) \quad I_{G(o_\ell)}(\sigma) = C_{G(o_\ell)}(x)K_\ell^m.$$

Indeed, let θ' be the unique extension of θ to $\mathfrak{r}_{\bar{x}}$ that corresponds to σ ; see Section 3.2. Choose $x \in \mathfrak{g}(o_\ell)$ such that $\varphi = \varphi_x \in \mathfrak{g}(o_\ell)^\vee$ satisfies $\varphi|_{\mathfrak{r}_{\bar{x}}} = \theta'$. Since x is regular we may apply Lemma 3.3 with $r = m$ and $\theta = \varphi|_{\mathfrak{g}(\pi^{m+1} o_\ell)}$, and get

$$I_{G(o_\ell)}(\sigma) \subset I_{G(o_\ell)}(\log^* \theta) = C_{G(o_\ell)}(x)K_\ell^m.$$

For the reverse inclusion recall that θ' determines σ and vice versa, therefore

$$I_{G(o_\ell)}(\sigma) \supset I_{I_{G(o_\ell)}(\log^* \theta)}(\sigma) = I_{I_{G(o_\ell)}(\log^* \theta)}(\log^* \theta') \supset C_{G(o_\ell)}(x)K_\ell^m,$$

and (3.1) follows; we remark that the last inclusion, which *a posteriori* is equality, follows from the fact that $C_{G(o_\ell)}(x)$ stabilises φ_x and in particular it stabilises its restriction θ' , and the fact that K_ℓ^m stabilises θ' by the Heisenberg lift construction.

We now show that σ extends to its inertia group $I_{G(o_\ell)}(\sigma)$. By Corollary (11.31) in [9] it is enough to show that σ extends to all the subgroups Q such that Q/K_ℓ^m is a Sylow subgroup of

and is therefore an extension of σ .

It follows that σ extends to its inertia group and induces irreducibly to $\mathbf{G}(\mathfrak{o}_\ell)$, proving the first assertion. To prove the second assertion we note that the possible extensions σ' of σ are in bijection with $C_{\mathbf{G}(\mathfrak{o}_\ell)}(x)/K_\ell^m \cap C_{\mathbf{G}(\mathfrak{o}_\ell)}(x) \simeq C_{\mathbf{G}(\mathfrak{o}_m)}(x_m)$. \square

Remark 3.5. We note that the order of choices made in the proof of Theorem 3.1 is crucial. We start with σ , which determines θ' , and only then choose x , which determines the extension $\varphi = \varphi_x$. The choice of x determines the decomposition of $I_{G(\mathfrak{o}_\ell)}(\sigma)$ as a product of the centraliser of x and the congruence subgroup and, in particular, determines P_x . Only then j_x is chosen as a submodule fixed by P_x .

Porism 3.6. *Under the additional assumption that $p = \text{char}(k) \nmid n$, Theorem 3.1 holds for \mathbf{SL}_n and \mathbf{SU}_n .*

Proof. If $(p, n) = 1$ the determinant map $\det : K_\ell^j \rightarrow (1 + \pi^j \mathfrak{D}_\ell)$ splits for every $1 \leq j \leq \ell$, since K_ℓ^1 is a p -group and the restriction of the determinant map to scalar matrices is the invertible map $aI \mapsto a^n$. It follows that

$$K_\ell^j \simeq (K'_\ell)^j \times \{\text{scalars}\},$$

where $(K'_\ell)^j = K_\ell^j \cap \mathbf{SL}_n(\mathfrak{D}_\ell)$.

Consequently $\sigma' := \text{Res}_{(K'_\ell)^m}^{\mathfrak{K}_\ell^m}(\sigma)$ is irreducible and all irreducible representations of $(K'_\ell)^m$ that lie above a regular linear character are obtained in this way. Moreover

$$I_{G'(\mathfrak{o}_\ell)}(\sigma') = I_{G(\mathfrak{o}_\ell)}(\sigma) \cap \mathbf{SL}_n(\mathfrak{D}_\ell),$$

hence the extendability of σ' to its inertia follows from the extendability of σ to its inertia. \square

Remark 3.7. Recall that the character σ appearing in Theorem 3.1 extends to its inertia group if and only if the associated two-cocycle $c(\sigma) \in H^2(I_{G(\mathfrak{o}_\ell)}(\sigma)/K_m^\ell, \mathbb{C}^\times)$ is trivial; see [9, Theorem 11.7]. In the case of \mathbf{GL}_n this two-cocycle coincides with the one appearing in [20, Hypothesis 3.2.1]. The first assertion of Theorem 3.1 therefore validates [20, Hypothesis 3.2.1] and [20, Conjecture 4.6.5] for arbitrary regular matrices, provided that the residual characteristic is odd.

4. REGULAR ELEMENTS IN $\mathfrak{g}(k)$

In this section there is no restriction on the characteristic of k . Let \bar{k} be an algebraic closure of k . For $d \in \mathbb{N}$, let $k_d \subset \bar{k}$ denote the unique extension of k of degree d . Let $\nu \in \text{Gal}(\bar{k}/k)$ be the Frobenius automorphism $\nu(x) = x^q$. We maintain the notation \circ for the restriction of ν to k_2 and its extension to an involution of $M_n(k_2)$ as a k -algebra.

4.1. Description and enumeration of regular elements in $\mathfrak{g}(k)$. Let \sim be the multiplicative involution on $k_2[t]$ defined by

$$h(t) = \sum_i c_i t^i \mapsto \tilde{h}(t) = \sum_i (-1)^{\deg(h)-i} c_i^\circ t^i.$$

By [2, Lemma 3.5] a regular matrix in $M_n(k_2)$ is similar to an anti-hermitian matrix if and only if its characteristic polynomial is fixed under this involution. Let

$$\mathcal{P}_{\text{gl}}(k_d) = \{f \in k_d[t] \mid f \text{ monic and irreducible}\}, \text{ and}$$

$$\mathcal{P}_{\text{gu}}(k) = \{f \in k_2[t] \mid f \in \mathcal{P}_{\text{gl}}(k_2) \text{ and } \tilde{f} = f, \text{ or } f = g \cdot \tilde{g} \text{ with } g \neq \tilde{g} \in \mathcal{P}_{\text{gl}}(k_2)\}.$$

Let $\mathcal{M}_{\mathfrak{g}}(k)$ be the multiplicative monoid generated by $\mathcal{P}_{\mathfrak{g}}(k)$. It follows that for every $a \in \mathfrak{g}(k)$, its characteristic polynomial Δ_a can be written uniquely (up to reordering) in the form $\Delta_a = \prod_i f_i^{e_i}$, for some finite list of elements $f_i \in \mathcal{P}_{\mathfrak{g}}(k)$ with $\sum_i \deg(f_i)e_i = n$, namely, that $\Delta_a \in \mathcal{M}_{\mathfrak{g}}(k)$.

Definition 4.1. We call a matrix $\tau \in M_n(\mathbb{Z}_{\geq 0})$ n -typical if $\sum_{d,e} de\tau_{d,e} = n$. We denote the set of n -typical matrices by \mathcal{A}_n .

For $h \in \mathcal{M}_{\mathfrak{g}}(k)$ of degree n let $\tau_{d,e}(h)$ be the number of distinct polynomials $f \in \mathcal{P}_{\mathfrak{g}}(k)$ occurring in h such that $d = \deg(f)$ and e is the multiplicity of f in h . We get an n -typical matrix which we call the *type* of h . For $\tau \in \mathcal{A}_n$ let $\mathcal{M}_{\mathfrak{g}}(k)^\tau$ be the subset of polynomials in $\mathcal{M}_{\mathfrak{g}}(k)$ of type τ . Let $w_d(x)$ be as in (1.1).

Proposition 4.2. Let $n \in \mathbb{N}$ and assume that $|k| = q > n$. For any $\tau \in \mathcal{A}_n$ we have

$$|\mathcal{M}_{\mathfrak{g}}(k)^\tau| = \prod_d \binom{\sum_e \tau_{d,e}}{\tau_{d,1}, \tau_{d,2}, \dots} \binom{w_d(q)}{\sum_e \tau_{d,e}}.$$

In particular $|\mathcal{M}_{\text{gu}}(k)^\tau| = |\mathcal{M}_{\text{gl}}(k)^\tau|$.

To prove Proposition 4.2 we require the following Lemma which is the anti-hermitian analogue of [4, Lemma 2].

Lemma 4.3. Let $f(t) \in k_2[t]$ be irreducible of degree d with $\tilde{f}(t) = f(t)$. Then d is odd and any root ξ of $f(t)$ is in the kernel of the trace map $\text{Tr}_{k_{2d}|k_d} : k_{2d} \rightarrow k_d$.

Proof. Assume first that q is odd. Consider the \circ -map on $R = M_d(k_2)$. For every matrix $b \in R$ we get an involution of k -algebras $\circ : C_R(b) \rightarrow C_R(b^\circ)$. By [2, Lemma 3.5] there exists $\xi \in R$ whose characteristic polynomial is $f(t)$ and $\xi + \xi^\circ = 0$. This, combined with the assumption that f is irreducible, means that $C_R(\xi) = C_R(\xi^\circ) = k_2[\xi] \simeq k_{2d}$. We thus have the following commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\circ} & R \\ \cup & & \cup \\ C_R(\xi) & \longrightarrow & C_R(\xi) \end{array}$$

and the restriction of \circ to $C_R(\xi)$ is a field automorphism of order two fixing k and hence of the form $\eta \mapsto \eta^{q^d}$ for all $\eta \in C_R(\xi)$. It follows that $\text{Tr}_{k_{2d}|k_d}(\xi) = \xi + \xi^{q^d} = \xi + \xi^\circ = 0$.

It remains to show that d is odd. We use the unitary version [4, Lemma 2] and the Cayley map. Recall the Cayley map $\text{Cay} : \mathfrak{gu}_d \rightarrow \mathbf{GU}_d$ defined by $a \mapsto (1+a)(1-a)^{-1}$. This is a rational map defined only on elements a that do not have -1 as an eigenvalue. Nevertheless, on elements of interest to us, namely generators of k_{2d} as an k_2 -algebra, it is always defined.

Observing that Cay commutes with the Galois action we deduce that ξ generates k_{2d} over k_2 if and only if $\text{Cay}(\xi)$ does. The fact that d is odd now follows from the unitary case.

Assume now that q is even. Recall that on k_2 the Galois action is $x^\circ = x^q$. The condition $f(t) = \tilde{f}(t)$ implies that both ξ and ξ^q are roots of $f(t)$. All the roots of the irreducible polynomial $f(t)$ are in the same orbit under the group generated by $x \rightarrow x^{q^2}$ and therefore $\xi^{q^{2i}} = \xi^q$ for some i satisfying $1 \leq i < d$, where $d = \deg(f)$. This, along with the fact that the exponent of ξ is coprime to q , implies that $\xi^{q^{2i-1}-1} = 1$. As ξ is a root of an irreducible polynomial of degree d we also have $\xi^{q^{2d}-1} = 1$. Let e be the exponent of ξ . We have

$$(4.1) \quad q^{2d} \equiv 1 \pmod{e} \quad \text{and} \quad q^{2i-1} \equiv 1 \pmod{e}.$$

Let $r = \gcd(2i-1, 2d)$. Since r is odd it divides d and therefore we get $q^d \equiv 1 \pmod{e}$. This implies that $\xi^{q^d-1} = 1$ and hence $\xi \in k_d$. It follows that d is odd as otherwise ξ lies in a proper subfield of k_{2d} that contains k_2 , in contradiction to ξ being a root of degree d irreducible polynomial over k_2 . Finally, we have $\text{tr}_{k_{2d}|k_d}(\xi) = \xi^{q^d} + \xi = 2\xi = 0$. This completes the proof for q even. \square

Proof of Proposition 4.2. First, assume that $\tau_{n,1} = 1$ and all other entries are zero. For $\mathfrak{g} = \mathfrak{gl}$ this means that τ corresponds to an irreducible polynomial of degree n . It is well known that the number of such polynomials over k is $w_n(q)$, that is, $|\mathcal{M}_{\mathfrak{gl}}(k)^\tau| = w_n(q)$. For the unitary counterpart we consider the odd and even cases separately. If n is odd we need to count the number of monic irreducible polynomials of degree n over k_2 that satisfy $\tilde{f}(t) = f(t)$. By Lemma 4.3 this amounts to counting the number of elements in the kernel of the trace map $\text{Tr}_{k_{2n}|k_n}$ excluding elements in the kernels $\text{Tr}_{k_{2m}|k_m}$ for $m|n$, and divide by n . Thus, as in the derivation of $w_n(q)$, an inclusion-exclusion argument gives that $|\mathcal{M}_{\mathfrak{gu}}(k)^\tau| = w_n(q)$. If n is even we need to count the number of polynomials $f(t) = g(t)\tilde{g}(t)$ with $g(t)$ and $\tilde{g}(t)$ distinct irreducible polynomials of degree $n/2$ over k_2 . In other words, we look for half the number of polynomials of degree $n/2$ over k_2 which are not fixed by the involution $g \mapsto \tilde{g}$. The total number of irreducible polynomials of degree $n/2$ over k_2 is $w_{\frac{n}{2}}(q^2)$. The number of irreducible polynomials over k_2 of degree $n/2$ which are fixed by the involution depends on the parity of $n/2$: it is zero if $n/2$ is even (by Lemma 4.3) and equals $w_{\frac{n}{2}}(q)$ if $n/2$ is odd. It follows that

$$|\mathcal{M}_{\mathfrak{gu}}(k)^\tau| = \begin{cases} \frac{1}{2}(w_{\frac{n}{2}}(q^2) - w_{\frac{n}{2}}(q)) & \text{if } n \equiv_4 2, \\ \frac{1}{2}w_{\frac{n}{2}}(q^2) & \text{if } n \equiv_4 0. \end{cases}$$

Breaking $w_n(q)$ to a sum over the even divisors and a sum over the odd divisors gives

$$w_n(q) = \frac{1}{n} \sum_{m|n} \mu(n/m) q^m = \frac{1}{2} \left(\frac{1}{n/2} \sum_{\substack{m|n \\ m \text{ even}}} \mu(n/m) q^m + \frac{1}{n/2} \sum_{\substack{m|n \\ m \text{ odd}}} \mu(n/m) q^m \right).$$

The first summand equals $w_{\frac{n}{2}}(q^2)$ as can be seen by substituting $m = 2j$ and $n = 2 \cdot \frac{n}{2}$. The second summand vanishes if n is divisible by 4 = 2^2 , and equals $-w_{\frac{n}{2}}(q)$ if $n/2$ is odd by the definition of the Möbius function. Thus $|\mathcal{M}_{\mathfrak{gu}}(k)^\tau| = w_n(q)$ for n even.

Finally, the formula for general τ now follows by multiplying over the various possibilities for the irreducible constituents and making sure that there are no repeated polynomials except for those dictated by τ . \square

4.2. Centralisers in $\mathbf{G}(k)$ of regular elements of $\mathfrak{g}(k)$. The following proposition describes the structure of centralisers of regular elements. The type of a regular element in $\mathfrak{g}(k)$ is the type of its characteristic polynomial. We say that a type $\tau \in \mathcal{A}_n$ is (d, e) -primary if $\tau_{d,e} = 1$ for d, e with $de = n$ and zero otherwise. That is, the characteristic polynomial of a is of the form $\Delta_a(t) = f(t)^e$ with $f(t) \in \mathcal{P}_{\mathfrak{g}}(k)$ of degree d . Let $\mathbf{H}_{\epsilon,d}$ be the following algebraic group

$$\mathbf{H}_{\epsilon,d} = \begin{cases} \mathbf{GL}_1, & \text{if } \epsilon = 1 \text{ (independent of } d), \text{ or } \epsilon = -1 \text{ and } d \text{ is even,} \\ \mathbf{GU}_{1,k_{2d}/k_d}, & \text{if } \epsilon = -1 \text{ and } d \text{ is odd,} \end{cases}$$

where $\mathbf{GU}_{1,k_{2d}/k_d}$ denotes the one-dimensional unitary group associated to the quadratic extension k_{2d}/k_d .

Proposition 4.4. *Let $C_{\mathbf{G}(k)}(a)$ denote the centraliser in $\mathbf{G}(k)$ of a regular element $a \in \mathfrak{g}(k)$ of type τ . Then*

$$C_{\mathbf{G}(k)}(a) \simeq \prod_{d,e} \mathbf{H}_{\epsilon_{\mathbf{G}},d} (k_d[t]/(t^e))^{\tau_{d,e}}.$$

In particular, the cardinality of the centraliser is $u_{\epsilon_{\mathbf{G}}}^{\tau}(q)$.

Proof. Let $a \in \mathfrak{g}(k)$ be regular of type τ . Assume first that τ is (d, e) -primary and write $\Delta_a(t) = f(t)^e$ with $f(t) \in \mathcal{P}_{\mathfrak{g}}(k)$ of degree d . If $\mathfrak{g} = \mathfrak{gl}_n$ then f is irreducible over k and $C_{\mathbf{M}_n(k)}(a) = k[a] \simeq k_d[u]/(u^e)$ and the result follows by taking units.

If $\mathfrak{g} = \mathfrak{gu}_n$ then f is of degree d over k_2 and satisfies $f = \tilde{f}$. Assume first that d is odd which means that f is irreducible. Write $a = S + N$, the additive Jordan decomposition of a , with S semisimple and N nilpotent such that $[S, N] = 0$. Then we have that $S, N \in \mathfrak{g}(k)$, and $C_{\mathbf{M}_n(k_2)}(a) = k_2[a] = k_2[S][N] \simeq k_{2d}[u]/(u^e)$. A short computation shows that intersecting the latter with $\mathbf{G}(k)$ gives $C_{\mathbf{G}(k)}(a) \simeq \mathbf{GU}_{1,k_{2d}/k_d}(k[u]/(u^e))$. Now assume that d is even and so f is a product of two distinct irreducible polynomials g and \tilde{g} of degree $d/2$. Let $\alpha \in \mathbf{M}_{d/2}(k_2)$ be a root of $g(t)$ so that $k_d \simeq k_2[\alpha] \hookrightarrow \mathbf{M}_{d/2}(k_2)$. Then $-\alpha^\circ$ is a root of $\tilde{g}(t)$ and a is similar in $\mathbf{M}_2(\mathbf{M}_e(\mathbf{M}_{d/2}(k_2))) = \mathbf{M}_n(k_2)$ to the block diagonal matrix $\text{diag}(J_e(\alpha), -J_e(\alpha)^\circ)$, where

$$J_e(\alpha) = \begin{pmatrix} \alpha & \mathbf{I} & & 0 \\ 0 & \alpha & \ddots & \\ \vdots & \ddots & \ddots & \mathbf{I} \\ 0 & \cdots & 0 & \alpha \end{pmatrix}.$$

A short computation shows that $C_{\mathbf{G}(k)}(a) \simeq \mathbf{GL}_1(k_d[u]/(u^e))$.

Finally, for general τ , if $\prod f_i(t)^{r_i}$ is a decomposition with distinct $f_i \in \mathcal{P}_{\mathfrak{g}}(k)$, then the centraliser of a is the product of the centralisers of the primary parts. \square

4.3. Centralisers in $\mathbf{G}(k) \cap \mathbf{SL}_n(k_2)$ of regular elements in $\mathfrak{g}(k)$. Let $\mathbf{G}' := \mathbf{G} \cap \mathbf{SL}_n$. Recall that for $\tau \in \mathcal{A}_n$, with associated characteristic polynomial $\prod_{i=1}^m f_i(t)^{r_i}$ such that $f_i \in \mathcal{P}_{\mathfrak{g}}(k)$ are distinct, we defined $\iota(\tau, q - \varepsilon) = \gcd(r_1, \dots, r_m, q - \varepsilon)$.

Proposition 4.5. *Let $a \in \mathfrak{g}(k)$ be a regular element of type $\tau \in \mathcal{A}_n$. Then*

$$|C_{\mathbf{G}(k)}(a) : C_{\mathbf{G}'(k)}(a)| = \frac{q - \varepsilon_{\mathbf{G}}}{\iota(\tau, q - \varepsilon_{\mathbf{G}})}.$$

Proof. Let \mathbf{G}_1 be either \mathbf{GL}_1 or \mathbf{GU}_1 according to \mathbf{G} being \mathbf{GL}_n or \mathbf{GU}_n , respectively. We have an exact sequence

$$(4.2) \quad 1 \rightarrow C_{\mathbf{G}'(k)}(a) \hookrightarrow C_{\mathbf{G}(k)}(a) \xrightarrow{\det} \mathbf{G}_1(k).$$

As $|C_{\mathbf{G}_1}(k)| = q - \varepsilon_{\mathbf{G}}$, we may equally well show that the size of the co-kernel of the determinant map is $\iota(\tau, q - \varepsilon_{\mathbf{G}})$. Suppose first that τ is $(n, 1)$ -primary. In such case the statement boils down to surjectivity of the rightmost arrow in (4.2). If $\mathfrak{g} = \mathfrak{gl}_n$ this means that τ corresponds to an irreducible polynomial of degree n over k . In such case $C_{\mathbf{G}(k)}(a) \simeq k_n^\times$ and the determinant map identifies with the norm map $N_{k_n|k} : k_n^\times \rightarrow k^\times$ which is surjective. If $\mathfrak{g} = \mathfrak{gu}_n$ we consider the two cases according to the parity of n . If n is odd then $C_{\mathbf{GU}_n(k)}(a) \simeq \ker N_{k_{2n}|k_n}$ and the determinant map identifies with the norm map $N_{k_{2n}|k_2}$. Its surjectivity on $\mathbf{GU}_1(k) = \ker N_{k_{2n}|k}$ follows from the surjectivity of the norm maps and Hilbert 90. For n even the characteristic polynomial of a is a product $g(t)\tilde{g}(t)$, with $g(t) \neq \tilde{g}(t)$ irreducible polynomials of degree $n/2$ over k_2 . We have $C_{\mathbf{G}(k)}(a) \simeq k_n^\times$ and the determinant map coincides with $\alpha \mapsto \alpha \cdot \alpha^{-q}$ which is, again, a surjection on $\mathbf{GU}_1(k)$.

For general τ we get the following diagram

$$\begin{array}{ccccc} \prod_{d,e} \mathbf{H}_{\varepsilon_{\mathbf{G}},d}(k_d[t]/(t^e))^{\times \tau_{d,e}} & \simeq & C_{\mathbf{G}(k)}(a) & \subset & \mathbf{G}(k) \\ \eta \downarrow & & \downarrow & & \downarrow \det \\ \prod_{d,e} (\mathbf{G}_1(k)^e)^{\times \tau_{d,e}} & \xrightarrow{\phi} & \mathbf{G}_1(k)^{\gcd\{e \mid \exists d \tau_{d,e} \neq 0\}} & \subset & \mathbf{G}_1(k), \end{array}$$

where the map η is composition of reduction modulo the radical and component-wise application of appropriate norm maps; the map ϕ is the group multiplication. For every primary factor $f(t)^e$ of the characteristic polynomial $\Delta_a(t)$ of a , the image of the restriction of the determinant map to the corresponding block in $C_{\mathbf{G}(k)}(a)$ is the subgroup of e -powers $\mathbf{G}_1(k)^e$ in the cyclic group $\mathbf{G}_1(k)$. Going over all the possible exponents e that occur in τ we get that the image of the determinant map is $\mathbf{G}_1(k)^{\gcd\{e \mid \exists d \tau_{d,e} \neq 0\}}$, as required. \square

5. DIMENSIONS, ENUMERATION AND ENNOLA DUALITY

We are now in a position to prove the remaining theorems and the corollary stated in the introduction.

Proof of Theorem B. We use the proof of Theorem 3.1. Regular characters of $\mathbf{G}(\mathfrak{o})$ of level ℓ are characters of $\mathbf{G}(\mathfrak{o}_{\ell+1})$. Let χ be such character.

Assume first that $\ell + 1 = 2m$. Then χ is induced from a linear character of the stabiliser $I_{\mathbf{G}(\mathfrak{o}_{2m})}(\sigma) = C_{\mathbf{G}(\mathfrak{o}_{2m})}(x)K_{2m}^m$ that extends some linear character σ of K_{2m}^m and $x \in \mathfrak{g}(\mathfrak{o}_{2m})$, see Figure 2. The degree is therefore

$$\begin{aligned} \chi(1) &= [\mathbf{G}(\mathfrak{o}_{2m}) : I_{\mathbf{G}(\mathfrak{o}_{2m})}(\sigma)] = \frac{|\mathbf{G}(\mathfrak{o}_m)|}{|C_{\mathbf{G}(\mathfrak{o}_m)}(x_m)|} \\ (5.1) \quad &= \frac{q^{n^2(m-1)}|\mathbf{G}(k)|}{q^{n(m-1)}|C_{\mathbf{G}(k)}(x_1)|} = q^{\binom{n}{2}(\ell-1)} \frac{v_{\varepsilon_{\mathbf{G}}}(q)}{u_{\varepsilon_{\mathbf{G}}}^{\tau}(q)}, \end{aligned}$$

where x_j denotes the reduction of x modulo π^j and τ is the type of x_1 .

If $\ell + 1 = 2m + 1$, the construction differs by a Heisenberg lift, that is, χ is induced from a character of $I_{\mathbf{G}(\mathfrak{o}_{\ell+1})}(\sigma)$ of degree $q^{\binom{n}{2}}$, yielding

$$\begin{aligned} \chi(1) &= q^{\binom{n}{2}} [\mathbf{G}(\mathfrak{o}_{\ell+1}) : I_{\mathbf{G}(\mathfrak{o}_{\ell+1})}(\sigma)] = q^{\binom{n}{2}} \frac{|\mathbf{G}(\mathfrak{o}_m)|}{|C_{\mathbf{G}(\mathfrak{o}_m)}(x_m)|} \\ (5.2) \quad &= q^{\binom{n}{2}(2m-2+1)} \frac{|\mathbf{G}(k)|}{|C_{\mathbf{G}(k)}(x_1)|} = q^{\binom{n}{2}(\ell-1)} \frac{v_{\varepsilon_{\mathbf{G}}}(q)}{u_{\varepsilon_{\mathbf{G}}}^{\tau}(q)}. \end{aligned}$$

□

Proof of Theorem C. Let $\tau \in \mathcal{A}_n$ be a type. By Proposition 4.2 the number of distinct polynomials of type τ is

$$|\mathcal{M}_{\mathfrak{g}}(k)^{\tau}| = \prod_d \binom{\sum_e \tau_{d,e}}{\tau_{d,1}, \tau_{d,2}, \dots} \binom{w_d(q)}{\sum_e \tau_{d,e}}.$$

It follows that this is the number of $\mathbf{G}(k)$ -orbits of regular elements in $\mathfrak{g}(k)$ of type τ , and hence also the number of $\mathbf{G}(k)$ -orbits of regular characters in $\mathfrak{g}(k)^{\vee}$ of type τ , see Section 2.3.

Assume first that $\ell + 1 = 2m$. Characters in each of these orbits, viewed as characters of $K^{\ell}/K^{\ell+1}$, extend to $K^m/K^{\ell+1}$. By regularity, the number of inequivalent extensions of each character of $K^{\ell}/K^{\ell+1}$ is $q^{n(\ell-m)}$. By Theorem 3.1 they all extend further to the stabiliser in

$$|I_{\mathbf{G}(\mathfrak{o}_{\ell+1})}(\sigma)/K_{\ell+1}^m| = q^{n(m-1)} u_{\varepsilon_{\mathbf{G}}}^{\tau}(q)$$

different ways, and they all induce to inequivalent irreducible character. It follows that the number of regular representations of $\mathbf{G}(\mathfrak{o}_{\ell+1})$ of type τ is

$$\begin{aligned} \text{Irr}(\mathbf{G}(\mathfrak{o}_{\ell+1}); \tau) &= q^{n(m-1)} \cdot |I_{\mathbf{G}(\mathfrak{o}_{\ell+1})}(\sigma)/K_{\ell+1}^m| \cdot |\mathcal{M}_{\mathfrak{g}}(k)^{\tau}| \\ (5.3) \quad &= q^{n(\ell-1)} u_{\varepsilon_{\mathbf{G}}}(q) \prod_d \binom{\sum_e \tau_{d,e}}{\tau_{d,1}, \tau_{d,2}, \dots} \binom{w_d(q)}{\sum_e \tau_{d,e}}. \end{aligned}$$

Similar considerations give the formula for $\ell + 1 = 2m + 1$ odd. The only twist is that one needs to take into account the q^n inequivalent extensions of linear characters of $K_{\ell+1}^{m+1}$ to $R_{\bar{x}}$, see Figure 3. □

Proof of Corollary D. Combining Theorems B and C gives

$$\begin{aligned} \zeta_{\mathbf{G}(\mathfrak{o}_{\ell+1})}^{\text{reg}}(s) &= \sum_{\tau \in \mathcal{A}_n} \sum_{\chi \in \text{Irr}(\mathbf{G}(\mathfrak{o}_{\ell+1}); \tau)} \chi(1)^{-s} \\ &= q^{(\ell-1)(n-\binom{n}{2}s)} \sum_{\tau \in \mathcal{A}_n} u_{\varepsilon_{\mathbf{G}}}^{\tau}(q) \prod_d \left(\frac{\sum_e \tau_{d,e}}{\tau_{d,1}, \tau_{d,2}, \dots} \right) \left(\frac{w_d(q)}{\sum_e \tau_{d,e}} \right) \frac{v_{\varepsilon_{\mathbf{G}}}(q)^{-s}}{u_{\varepsilon_{\mathbf{G}}}^{\tau}(q)^{-s}}. \end{aligned}$$

The corollary follows by $\zeta_{\mathbf{G}(\mathfrak{o})}^{\text{reg}}(s) = \sum_{\ell=1}^{\infty} \zeta_{\mathbf{G}(\mathfrak{o}_{\ell+1})}^{\text{reg}}(s)$. \square

Proof of Theorem E. By Porism 3.6 the construction theorem (Theorem 3.1) continues to hold upon replacing \mathbf{G} by \mathbf{G}' . To get the regular representation zeta function of $\mathbf{G}'(\mathfrak{o})$ we follow the proofs of Theorems B and C and replace the occurrences of \mathbf{G} by \mathbf{G}' and of \mathfrak{g} by $\mathfrak{g}' := \mathfrak{g} \cap \mathfrak{sl}_n$. We now address the main points that need further justification.

First, the occurrences of $|\mathbf{G}(k)|/|C_{\mathbf{G}(k)}(x_1)|$ in equations (5.1) and (5.2) are replaced by $|\mathbf{G}'(k)|/|C_{\mathbf{G}'(k)}(x_1)|$. To justify the latter we need to show that the restriction of the reduction map $\mathbf{G}(\mathfrak{o}_m) \rightarrow \mathbf{G}(k)$ to

$$(5.4) \quad C_{\mathbf{G}'(\mathfrak{o}_m)}(x) \rightarrow C_{\mathbf{G}'(k)}(x_1)$$

is surjective. Since the map $C_{\mathbf{G}(\mathfrak{o}_m)}(x) \rightarrow C_{\mathbf{G}(k)}(x_1)$ is surjective, for every $g_1 \in C_{\mathbf{G}'(k)}(x_1)$ there exists a preimage $g \in C_{\mathbf{G}(\mathfrak{o}_m)}(x)$. This means that $\det(g) \in 1 + \pi \mathfrak{D}_m$. Since p is prime to n we can find a scalar matrix $aI \in K_m^1$ with $\det(g) = \det(aI)$ so that $g' = ga^{-1}I \in C_{\mathbf{G}'(\mathfrak{o}_m)}(x)$ and $g' \equiv_{\pi} g_1$, hence the surjectivity of the reduction map (5.4).

Therefore, consulting Proposition 4.5, we deduce that the degree of a regular character χ of $\mathbf{G}'(\mathfrak{o}_{\ell+1})$ lying above characters of type τ is

$$\chi(1) = q^{\binom{n}{2}(\ell-1)} \frac{v_{\varepsilon_{\mathbf{G}}}(q)}{u_{\varepsilon_{\mathbf{G}}}^{\tau}(q) \iota(\tau, q - \varepsilon_{\mathbf{G}})}.$$

Second, we address the enumeration of such characters. The trace form on $M_n(k_2)$ restricts to a non-degenerate pairing between $\mathfrak{g}'(k)$ and $\mathfrak{g}'(k)^{\vee}$ as long as we require that the characteristic is prime to n . We then get an equivariant isomorphism between the two. The well-defined surjective map $\mathfrak{g}(k) \rightarrow \mathfrak{g}'(k)$ given by $a \mapsto a - \frac{1}{n} \text{Tr}(a) I_n$, implies that the types that occur in $\mathfrak{g}'(k)$ are the same as those occurring in $\mathfrak{g}(k)$, however, the number of classes of every type τ is reduced by a factor of q , namely

$$(5.5) \quad |\mathcal{M}_{\mathfrak{g}'}(k)^{\tau}| = q^{-1} |\mathcal{M}_{\mathfrak{g}}(k)^{\tau}|.$$

We also note that each $\mathbf{G}(k)$ -class in $\mathfrak{g}'(k)$ splits into $\iota(\tau, q - \varepsilon_{\mathbf{G}})$ distinct $\mathbf{G}'(k)$ -classes of equal cardinality since for every $a \in \mathfrak{g}(k)$ we have

$$\frac{|\mathbf{G}(k)a|}{|\mathbf{G}'(k)a|} = \frac{[\mathbf{G}(k) : C_{\mathbf{G}(k)}(a)]}{[\mathbf{G}'(k) : C_{\mathbf{G}'(k)}(a)]} = \iota(\tau, q - \varepsilon_{\mathbf{G}}).$$

The number of such regular characters is given by the following \mathbf{G}' -analogue of (5.3)

$$\begin{aligned} \text{Irr}(\mathbf{G}'(\mathfrak{o}_{\ell+1}); \tau) &= q^{(n-1)(m-1)} \cdot |I_{\mathbf{G}'(\mathfrak{o}_{\ell+1})}(\sigma)/(K')_{\ell+1}^m| \cdot |\mathcal{M}_{\mathbf{g}'}(k)^\tau| \\ &= q^{(n-1)(\ell-1)} u_{\varepsilon_{\mathbf{G}}}(q) \frac{\iota(\tau, q - \varepsilon_{\mathbf{G}})}{q - \varepsilon_{\mathbf{G}}} \cdot q^{-1} \prod_d \left(\frac{\sum_e \tau_{d,e}}{\tau_{d,1}, \tau_{d,2}, \dots} \right) \left(\frac{w_d(q)}{\sum_e \tau_{d,e}} \right). \end{aligned}$$

Consequently, the regular zeta function of $\mathbf{G}'(\mathfrak{o}_{\ell+1})$ is

$$\begin{aligned} \zeta_{\mathbf{G}'(\mathfrak{o}_{\ell+1})}^{\text{reg}}(s) &= \sum_{\tau \in \mathcal{A}_n} \iota(\tau, q - \varepsilon_{\mathbf{G}}) \sum_{\chi \in \text{Irr}(\mathbf{G}'(\mathfrak{o}_{\ell+1}); \tau)} \chi(1)^{-s} \\ &= q^{(\ell-1)(n-1-\binom{n}{2}s)} \sum_{\tau \in \mathcal{A}_n} \frac{\iota(\tau, q - \varepsilon_{\mathbf{G}})^2}{(q - \varepsilon_{\mathbf{G}})} u_{\varepsilon_{\mathbf{G}}}^\tau(q) \\ &\quad \times q^{-1} \prod_d \left(\frac{\sum_e \tau_{d,e}}{\tau_{d,1}, \tau_{d,2}, \dots} \right) \left(\frac{w_d(q)}{\sum_e \tau_{d,e}} \right) \left(\frac{v_{\varepsilon_{\mathbf{G}}}(q)}{u_{\varepsilon_{\mathbf{G}}}^\tau(q) \iota(\tau, q - \varepsilon_{\mathbf{G}})} \right)^{-s}, \end{aligned}$$

and finally, we have $\zeta_{\mathbf{G}'(\mathfrak{o})}^{\text{reg}}(s) = \sum_{\ell=1}^{\infty} \zeta_{\mathbf{G}'(\mathfrak{o}_{\ell+1})}^{\text{reg}}(s)$. \square

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